EXTENSIONS OF VERMA MODULES

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Fix a finite dimensional semisimple Lie algebra over $\mathbf{C}$, and a Borel subalgebra. It is a long standing problem to determine the extension groups between Verma modules in the category $\mathcal{O}$ of [BGG]. No general formula, even conjectural, is known.

Let W be the Weyl group and $w_0 \in W$ the longest element. For $w \in W$ write Δ_w for the Verma module of highest weight $w^{-1}w_0 \cdot 0$ (dot action). The purpose of this note is to give a recursive formula (Corollary 5) for $\dim \mathrm{Ext}^1(\Delta_v, \Delta_w)$ (throughout '$\mathrm{Ext}^\bullet$' = '$\mathrm{Ext}^\bullet_{\mathcal{O}}$').

Write X for the flag variety associated to our Lie algebra. For each $w \in W$, we have a Schubert cell $C_w \subseteq X$ and an opposite Schubert cell $C^w \subseteq X$. Let $\ell\colon W \to \mathbf{Z}_{\geq 0}$ denote the length function and $\leq$ the Bruhat order. If $v \leq w$, then $C^v \cap C_w$ is affine and smooth of dimension $\ell(w) - \ell(v)$ (see [R]). Further, $C^v \cap C_v = \mathrm{pt}$; and if $v \not\leq w$, then $C^v \cap C_w = \varnothing$.

By [RSW, Proposition 4.2.1] and [BGS, Proposition 3.5.1]):

$$\mathrm{Ext}^\bullet(\Delta_v, \Delta_w) = H_c^{\bullet+\ell(w)-\ell(v)}(C^v \cap C_w), \quad \text{for all } v, w \in W,$$

where H_c^* denotes compactly supported cohomology (with $\mathbf{C}$-coefficients). The cohomology $H_c^*(C^v \cap C_w)$ comes equipped with a canonical (rational) Hodge structure which is respected by the usual long exact sequences. Consequently, we can and will view the extensions $\mathrm{Ext}^\bullet(\Delta_v, \Delta_w)$ as Hodge structures. Denote the trivial 1-dimensional Hodge structure by $\mathbf{Q}^H$, and write (n) for the n-th Tate twist.

The following Proposition is attributed to V. Deodhar in [RSW] (see [RSW, Lemma 4.3.1] for a proof).

Proposition 1. *Let $v, w \in W$ with $v \leq w$. Let $s \in W$ be a simple reflection such that $ws < w$.*

(i) *If $vs < v$, then $C^v \cap C_w \simeq C^{vs} \cap C_{ws}$.*
(ii) *If $vs > v$ and $vs \not\leq ws$, then $C^v \cap C_w \simeq C^v \cap C_{ws} \times \mathbf{C}^*$.*
(iii) *If $vs > v$ and $vs \leq ws$, then there exists a closed immersion*

$$(C^{vs} \cap C_{ws}) \times \mathbf{C} \hookrightarrow C^v \cap C_w$$

with open complement isomorphic to $(C^v \cap C_{ws}) \times \mathbf{C}^*$.

Corollary 2. $\mathrm{Hom}(\Delta_v, \Delta_w)$ *is pure of weight* 0.

Proof. It is known that $\dim \operatorname{Hom}(\Delta_x, \Delta_y) = 1$ if and only if $x \leq y$, and that $\operatorname{Hom}(\Delta_x, \Delta_y) = 0$ if $x \not\leq y$. In particular, we may assume $v \leq w$ and proceed by induction on w. If $w = v$, the result is clear. Otherwise, pick a simple reflection s such that $ws < w$. In the situation of Proposition 1(i), the result is obvious. In the situation of Proposition 1(ii) and/or (iii), the long exact sequence of cohomology and Künneth formula yield an inclusion $\operatorname{Hom}(\Delta_v, \Delta_{ws}) \hookrightarrow \operatorname{Hom}(\Delta_v, \Delta_w)$. Since in both of these situations $v \leq ws$, this inclusion must be an isomorphism. □

Corollary 3. *Let $v, w \in W$ with $v \leq w$. Let $s \in W$ be a simple reflection such that $ws > w$.*

(i) *If $vs < v$, then $\operatorname{Ext}^1(\Delta_v, \Delta_{ws}) \simeq \operatorname{Ext}^1(\Delta_{vs}, \Delta_w)$.*
(ii) *If $vs > v$ and $vs \not\leq w$, then $\operatorname{Ext}^1(\Delta_v, \Delta_w) \oplus \mathbf{Q}^H(-1) \simeq \operatorname{Ext}^1(\Delta_v, \Delta_{ws})$.*
(iii) *If $vs > v$ and $vs \leq w$, then there is an exact sequence*

$$0 \to \mathbf{Q}^H(-1) \to \operatorname{Ext}^1(\Delta_v, \Delta_w) \oplus \mathbf{Q}^H(-1) \to \operatorname{Ext}^1(\Delta_v, \Delta_{ws}) \to \operatorname{Ext}^1(\Delta_{vs}, \Delta_w)(-1).$$

Proof. (i) is clear. Further, $\dim \operatorname{Hom}(\Delta_x, \Delta_y) = 1$ if and only if $x \leq y$. Thus, the Künneth formula yields (ii). In the situation of (iii):

$$\operatorname{Hom}(\Delta_v, \Delta_w) = \operatorname{Hom}(\Delta_v, \Delta_{ws}) = \operatorname{Hom}(\Delta_{vs}, \Delta_w) = \mathbf{Q}^H.$$

So the cohomology long exact sequence and Künneth formula yield (iii). □

Theorem 4. *$\operatorname{Ext}^1(\Delta_v, \Delta_w)$ is pure of weight 2.*

Proof. We may assume $v \leq w$. Proceed by downwards induction on w. If w is the longest element, this is [M, Theorem 32]. If w is not the longest element, pick a simple reflection s such that $ws > w$ and apply Corollary 3. □

Corollary 5. *Let $v, w \in W$ with $v \leq w$. Let $s \in W$ be a simple reflection such that $ws < w$. Then*

$$\dim \operatorname{Ext}^1(\Delta_v, \Delta_w) = \begin{cases} \dim \operatorname{Ext}^1(\Delta_{vs}, \Delta_{ws}) & \text{if } vs < v; \\ 1 + \dim \operatorname{Ext}^1(\Delta_v, \Delta_{ws}) & \text{if } vs > v \text{ and } vs \not\leq ws; \\ \dim \operatorname{Ext}^1(\Delta_v, \Delta_{ws}) & \text{if } vs > w \text{ and } vs \leq ws. \end{cases}$$

Some concluding observations:

(i) $\dim \operatorname{Ext}^1(\Delta_v, \Delta_w)$ coincides with $(-1)^{\ell(w)-\ell(v)}$ times the coefficient of q in the corresponding Kazhdan-Lusztig R-polynomial.
(ii) Theorem 4 implies that the graded algebra $\bigoplus_{v,w \in W} \operatorname{Ext}^\bullet(\Delta_v, \Delta_w)$ *cannot*, in general, be generated in degree 0 and 1. As otherwise $\operatorname{Ext}^i(\Delta_v, \Delta_w)$ would be pure of weight $2i$. This would contradict [Boe], since the Kazhdan-Lusztig R-polynomials are the Hodge-Euler polynomials of the $C^v \cap C_w$.

(iii) Using Corollary 5 one can check

$$\dim \operatorname{Ext}^1(\Delta_v, \Delta_w) = \dim \operatorname{Ext}^{\ell(w)-\ell(v)-1}(\Delta_v, \Delta_w).$$

This upgrades to a canonical isomorphism by combining Theorem 4 with the main result of [BGS].

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